

# LAGRANGIAN KLEIN BOTTLES IN $\mathbb{R}^{2n}$

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The  $n$ -dimensional Klein bottle  $K^n$ ,  $n \geq 2$ , is obtained by gluing the ends of the cylinder  $S^{n-1} \times [0, 1]$  via an orientation reversing isometry of the standard  $(n-1)$ -sphere  $S^{n-1} \subset \mathbb{R}^n$ .

**Theorem.** *The  $n$ -dimensional Klein bottle  $K^n$  admits a Lagrangian embedding into the standard symplectic  $2n$ -space  $(\mathbb{R}^{2n}, \omega_0)$  if and only if  $n$  is odd.*

The existence of Lagrangian embeddings of *odd*-dimensional Klein bottles into  $(\mathbb{R}^{2n}, \omega_0)$  was proved by Lalonde [10]. As observed in [14], an explicit embedding is suggested by Picard–Lefschetz theory. Indeed, the antipodal map

$$\mathbb{R}^{2k+1} \supset S^{2k} \ni (x_1, x_2, \dots, x_{2k+1}) \mapsto (-x_1, -x_2, \dots, -x_{2k+1}) \in S^{2k} \subset \mathbb{R}^{2k+1}$$

reverses the orientation on  $S^{2k} \subset \mathbb{R}^{2k+1}$  and therefore the formula

$$S^{2k} \times [0, 1] \ni (x_1, \dots, x_{2k+1}, t) \mapsto (e^{\pi i t} x_1, \dots, e^{\pi i t} x_{2k+1}) \in \mathbb{C}^{2k+1} \quad (1)$$

defines an embedding of the odd-dimensional Klein bottle  $K^{2k+1}$  into  $\mathbb{C}^{2k+1} = \mathbb{R}^{4k+2}$ . It is easy to check that this embedding is Lagrangian with respect to the standard symplectic form  $\omega_0 = \frac{i}{2} \sum dz_\ell \wedge d\bar{z}_\ell$ .

Thus, the main task is to prove that an *even*-dimensional Klein bottle does not admit a Lagrangian embedding into  $(\mathbb{R}^{2n}, \omega_0)$ . For the usual Klein bottle  $K^2$ , this problem was proposed by Givental’ [6] and resolved by Shevchishin [16]. The argument in the general case follows the lines of the author’s alternative proof of Shevchishin’s result [15] (cf. also [4]). Namely, self-linking invariants introduced by Rokhlin and Viro are used to show that a suitable Luttinger-type surgery along a Lagrangian  $K^{2k} \subset \mathbb{R}^{4k}$  would produce an impossible symplectic manifold.

**1. Rokhlin and Viro indices for totally real Klein bottles.** Let us fix  $n$  and denote the  $n$ -dimensional Klein bottle simply by  $K$ . Let  $m \subset K$  be a fibre of the natural fibre bundle  $K \rightarrow S^1$ . Then  $m$  is an embedded  $(n-1)$ -dimensional sphere in  $K$ . Note that  $m$  is co-orientable and choose a non-vanishing normal vector field  $\nu_{m,K}$  on  $m$ .

Consider now a totally real embedding  $K \hookrightarrow \mathbb{C}^n$ , i.e., an embedding such that  $T_p K$  is transversal to  $iT_p K$  at every point  $p \in K$ . (For a Lagrangian embedding, the subspaces  $T_p K$  and  $iT_p K$  would be orthogonal with respect to the standard metric on  $\mathbb{C}^n$ .) Let  $m^\sharp$  be the pushoff of  $m$  in the direction of the vector field  $i\nu_{m,K}$ . The mod 2 homology class  $[m^\sharp] \in H_{n-1}(\mathbb{C}^n \setminus K; \mathbb{Z}/2)$  is independent of the choice of  $\nu_{m,K}$ . The linking number

$$V = \text{lk}(K, m^\sharp) \in \mathbb{Z}/2$$

is called the *Viro index* of  $m \subset K$  (cf. [15], §1.2).

In order to compute  $V$ , we choose an immersed  $n$ -ball  $M = \iota(B^n) \subset \mathbb{C}^n$  such that

- a)  $\partial M = \iota(\partial B^n) = m$ , and  $M$  is normal to  $K$  along  $m$ ;
- b) the self-intersections of  $M$  and the intersections of its interior with  $K$  are transverse double points;
- c) the tangent (half-)space of  $M$  at a point  $p \in m = \partial M$  is spanned by  $T_p m$  and  $i\nu_{m,K}$ ;

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d) the  $\mathbb{RC}$ -singular points of  $M$  are generic (see the definitions in §3 below).

(Immersion satisfying (a) and (b) are called *membranes* spanned by  $m$ .) Note that by (c) the pushoff of  $m$  inside  $M$  is precisely  $m^\sharp$ , and hence

$$V = \#(M \cap K) \bmod 2, \quad (2)$$

where  $\#(M \cap K)$  denotes the number of interior intersection points of  $M$  and  $K$ .

Suppose now that  $n$  is even and consider the *Rokhlin index* of  $M$  defined by the formula

$$R = n(M, \nu_{m,K}) + \#(M \cap K), \quad (3)$$

where  $n(M, \nu_{m,K}) \in \mathbb{Z}$  is the obstruction to extending  $\nu_{m,K}$  to a non-vanishing normal vector field on  $M$ , i.e., the algebraic number of zeroes of a generic normal extension. (For odd  $n$ , this number is defined only mod 2.)

**Lemma 1.**  $R = 0 \bmod 2$ .

This will be proved in §2 using nothing much. Note, however, that this is the only place where the assumption that  $n$  is even will be used in a crucial way (see Remark 5).

**Lemma 2.**  $n(M, \nu_{m,K}) = 1 \bmod 2$ .

This will be proved in §4 using a topological count of  $\mathbb{RC}$ -singularities recalled briefly in §3 following Domrin [2].

**Lemma 3.**  $V = 1 \bmod 2$ .

This follows immediately from formulas (2) and (3) and the preceding two lemmas and will play a key role in the proof of the main theorem in §5 and §6.

**2. Proof of Lemma 1.** Cut  $K$  along  $m$  and glue two copies of  $M$  into the resulting ‘holes’ to obtain an  $n$ -sphere  $S$ . Choose an orientation on  $S$  and note that it induces the *same* orientation on each of the two copies of  $M$ . (If we had  $S^{n-1} \times S^1$  instead of the Klein bottle, the orientations would be opposite.) Let  $\nu$  be a generic normal extension of  $\nu_{m,K}$  to  $M$ . Transform  $S$  into a generically immersed sphere by pushing the two copies of  $M$  apart in the direction of  $\nu$  and then smoothing the result.

Now we can compute the normal Euler number of  $S$  and the algebraic number of its double points. Namely,

$$n(S) = n(K) + 2n(M, \nu_{m,K}) = 2n(M, \nu_{m,K}),$$

where we have used the fact that for a totally real embedding of  $K$  the normal Euler number is equal to the Euler characteristic of  $K$  which is zero. Similarly,

$$\#_{alg}(S) = n(M, \nu_{m,K}) + 2\#_{alg}(M \cap K) + 4\#_{alg}(M),$$

where the signs in  $\#_{alg}(M \cap K)$  and  $\#_{alg}(M)$  are given by the induced orientations on  $M$  and  $K$  as subsets of  $S$ .

On the other hand, by the usual formula for the homological self-intersection index of an *oriented* immersed submanifold, we have

$$[S] \cdot [S] = n(S) + 2\#_{alg}(S) = 4(n(M, \nu_{m,K}) + \#_{alg}(M \cap K) + 2\#_{alg}(M)).$$

The homology class  $[S]$  is obviously trivial in  $\mathbb{C}^n$ , hence

$$n(M, \nu_{m,K}) + \#_{alg}(M \cap K) + 2\#_{alg}(M) = 0,$$

and the result follows from (3) because  $\#_{alg}(M \cap K) = \#(M \cap K) \bmod 2$ .  $\square$

**Remark 4.** The above argument and the result for  $n = 2$  go back to Rokhlin (see [8] and the proof of Lemma 1.12 in [15]). Note that we are actually proving a congruence modulo 8 using a trivial case of van der Blij's lemma to conclude that  $[S] \cdot [S] = 0 \bmod 8$ .

**Remark 5.** For odd  $n$ , the residue  $R \bmod 2$  is well-defined but the lemma is false. (Our proof does not work because the intersection index is not symmetric.) For instance, for the embedding given by (1), the totally real  $n$ -ball  $\{(z_1, \dots, z_n) \in \mathbb{C}^n \mid z_j \in \mathbb{R}, \|z\| \leq 1\}$  is a membrane satisfying conditions (a)-(d) and such that  $R = 1 \bmod 2$ .

**3.  $\mathbb{RC}$ -singularities and characteristic classes.** Here's a digression needed for the proof of Lemma 2. The material is mostly taken from [2] (cf. also [9] and [17]).

Let  $j : N \rightarrow \mathbb{C}^n$  be an immersion of a real oriented  $n$ -dimensional manifold. (Note that the real dimension of  $N$  is equal to the complex dimension of  $\mathbb{C}^n$ .) A point  $p \in N$  is called  *$\mathbb{RC}$ -singular* if the dimension of the maximal complex subspace in  $j_*T_pN \subset \mathbb{C}^n$  is positive (i.e., larger than expected). This dimension is called the *order* of an  $\mathbb{RC}$ -singular point. Denote by  $C_\mu(N)$  the set of  $\mathbb{RC}$ -singular points of order  $\mu$  and by  $C(N)$  the set of all  $\mathbb{RC}$ -singular points.

Let  $j_*^\mathbb{C} : TN \otimes \mathbb{C} \rightarrow j^*T\mathbb{C}^n$  be the complex vector bundle map given by  $j_*^\mathbb{C}(v \otimes \lambda) := \lambda j_*(v)$ . Its kernel at a point  $p \in N$  is isomorphic to the maximal complex subspace of  $j_*T_pN$ . Thus,  $C_\mu(N)$  coincides with the singularity set  $\Sigma_\mu = \{p \in N \mid \text{rk}_\mathbb{C} j_*^\mathbb{C} = n - \mu\}$ . If the immersion  $j$  is generic, then by [2], Lemma 1.3, the bundle map  $j_*^\mathbb{C}$  is generic in the sense of [13]. Hence, each  $C_\mu(N)$  is an oriented  $(n - 2\mu^2)$ -dimensional submanifold,  $C(N) = \overline{C_1(N)}$ , and there exists a canonical desingularisation  $\tilde{\Sigma}_1 \rightarrow \tilde{\Sigma}_1 = C(N)$  such that the complex line bundle  $\text{Ker } j_*^\mathbb{C}|_{\Sigma_1}$  extends to  $\tilde{\Sigma}_1$ . (Explicitly,  $\tilde{\Sigma}_1$  is the closure of the image of  $\text{Ker } j_*^\mathbb{C}|_{\Sigma_1}$  in the projectivisation of  $TN \otimes \mathbb{C}$  and the extension of  $\text{Ker } j_*^\mathbb{C}|_{\Sigma_1}$  is given by the tautological line bundle.) The extended bundle lies in the kernel of the pull-back of  $j_*^\mathbb{C}$  to  $\tilde{\Sigma}_1$  and therefore corresponds to a complex line subbundle  $\mathcal{Z}$  of the pull-back of  $TN$ .

Assume now that the manifold  $N$  is compact without boundary and its dimension  $n$  is even. Define  $[C(N)]$  as the fundamental class of the oriented manifold  $\tilde{\Sigma}_1$ . Then from Theorem 3 and Remark 1.4 in [2] one obtains the formula

$$\langle c_1(\mathcal{Z})^{(n/2-1)}, [C(N)] \rangle = \langle c_{n/2}(-TN \otimes \mathbb{C}), [N] \rangle, \quad (4)$$

where  $-TN \otimes \mathbb{C}$  denotes the  $K$ -theoretic inverse of  $TN \otimes \mathbb{C}$ . In the statements of the results in [2] it is assumed that  $C(N) = C_1(N)$  but the proofs carry over to the general case with formal changes. ( $\tilde{\Sigma}_1$  has to be used instead of the set  $\Sigma$  introduced on p. 910 of [2].)

**Remark 6.** The  $\mathbb{RC}$ -singular points of a generic immersed surface in  $\mathbb{C}^2$  are isolated (and more often referred to as 'complex points' or 'complex tangencies'). Formula (4) reduces in this case to the elementary formula  $I_+ - I_- = 0$ , where the *Lai indices*  $I_\pm$  of an oriented immersed surface are defined by counting its complex points with suitable signs (see [2], §3).

**4. Proof of Lemma 2** (cf. [15], Proof of Lemma 1.13). Let us construct a normal extension of  $\nu_{m,K}$  to  $M$  in the following way. Consider the vector field  $i\nu_{m,K}$ . It is tangent to  $M$  and transverse to  $\partial M$  by the choice of  $M$ . Let  $\tau$  be an extension of this vector field to a tangent vector field on  $M$  with a single transverse zero. (Recall that  $M$  is a ball.) Then  $-i\tau$  gives a normal extension of  $\nu_{m,K}$  that vanishes at the zero of  $\tau$  and at the points where  $\tau$  lies in a non-trivial complex subspace contained in  $T_pM$ . For a sufficiently generic  $\tau$ , the latter points lie in  $C_1(M)$ . In other words, we have to count the zeroes of a (generic) section of the quotient bundle  $\mathcal{E} = \tilde{T}M/\mathcal{Z}$ , where  $\tilde{T}M$  is the pull-back of  $TM$  to  $\tilde{\Sigma}_1$ . As we only need the answer mod 2, it is given by the evaluation of the top Stiefel–Whitney class  $w_{n-2}(\mathcal{E})$  on the fundamental class  $[C(M)] := [\tilde{\Sigma}_1]$ .

Since  $TM$  is trivial, we have

$$1 = (1 + w_2(\mathcal{Z}))(1 + w_1(\mathcal{E}) + \cdots + w_{n-2}(\mathcal{E})) \quad (5)$$

by the Whitney formula. It follows immediately that

$$w_{n-2}(\mathcal{E}) = w_2(\mathcal{Z})^{(n/2-1)}$$

and hence

$$\langle w_{n-2}(\mathcal{E}), [C(M)] \rangle = \langle w_2(\mathcal{Z})^{(n/2-1)}, [C(M)] \rangle = \langle c_1(\mathcal{Z})^{(n/2-1)}, [C(M)] \rangle \pmod{2}.$$

In order to show that the latter quantity vanishes (already as an integer), we apply formula (4) to an immersed sphere  $S$  similar to the one used in the proof of Lemma 1 above. Namely, we glue two copies of  $M$  to  $K$  cut along  $m$  but this time only smoothen the result near  $m$ . Condition (c) in §1 ensures that this smoothing can be done so that no additional  $\mathbb{R}\mathbb{C}$ -singularities are created and hence the set  $C(S)$  consists of two copies of  $C(M)$  with the same orientation and the same line bundle  $\mathcal{Z}$ . Thus,

$$2\langle c_1(\mathcal{Z})^{(n/2-1)}, [C(M)] \rangle = \langle c_1(\mathcal{Z})^{(n/2-1)}, [C(S)] \rangle \stackrel{(4)}{=} \langle c_{n/2}(-TS \otimes \mathbb{C}), [S] \rangle = 0.$$

It follows that  $\langle w_{n-2}(\mathcal{E}), [C(M)] \rangle = 0 \pmod{2}$  and hence the normal projection of  $-i\tau$  has an odd number of zeroes, which proves that  $n(M, \nu_{m,K}) = 1 \pmod{2}$ .  $\square$

**Remark 7.** For odd  $n$ , the vanishing of  $w_{n-2}(\mathcal{E})$  follows already from (5) without any appeal to (4). Thus Lemma 2 is true in that case as well.

**5. Dehn surgery.** Let  $U \supset K$  be a tubular neighbourhood of a totally real embedded Klein bottle  $K \subset \mathbb{C}^n$ . We consider two distinguished classes in the homology group  $H_{n-1}(\partial U; \mathbb{Z}/2)$ . Firstly, the fibre class  $[\delta]$  generating the kernel of the inclusion homomorphism  $H_{n-1}(\partial U; \mathbb{Z}/2) \rightarrow H_{n-1}(\overline{U}; \mathbb{Z}/2)$  and, secondly, the class  $[m^\sharp]$  of the  $\mathbb{C}$ -normal pushoff of  $m$  introduced in §1.

**Lemma 8** (cf. [15], Theorem 2.2). *Consider a surgery  $X = \overline{U} \cup_f (\mathbb{C}^n \setminus U)$  defined by a diffeomorphism  $f : \partial U \rightarrow \partial U$  such that*

$$f_*[\delta] = [\delta] + [m^\sharp]. \quad (6)$$

*If  $n$  is even, then  $K$  is homologically non-trivial in  $X$ . In particular,  $H_n(X; \mathbb{Z}/2) \neq 0$ .*

*Proof.* As  $n$  is even, we know that  $\text{lk}(K, m^\sharp) = 1 \pmod{2}$  by Lemma 3 and the definition of the Viro index. Since  $\text{lk}(K, \delta) = 1 \pmod{2}$  by definition, it follows that the sum  $[\delta] + [m^\sharp]$  bounds a mod 2 chain in  $\mathbb{C}^n - U$ . By property (6), this chain and the  $n$ -ball bounded by  $\delta$  in  $\overline{U}$  are glued into a mod 2 cycle in  $X$  whose intersection index with  $K$  is 1 mod 2.  $\square$

**Lemma 9.** *If  $X$  is orientable, then  $H_2(X; \mathbb{R}) = 0$ .*

*Proof.* Note first that  $H_2(X; \mathbb{R}) = H_c^{2n-2}(X; \mathbb{R})$  by Poincaré(-Lefschetz) duality. Since  $X \setminus K = \mathbb{C}^n \setminus K$ , an inspection of the cohomology long exact sequences

$$\cdots \rightarrow H_c^{2n-3}(K; \mathbb{R}) \rightarrow H_c^{2n-2}(\mathbb{C}^n \setminus K; \mathbb{R}) \rightarrow H_c^{2n-2}(\mathbb{C}^n; \mathbb{R}) \cong 0$$

$$\cdots \rightarrow H_c^{2n-2}(X \setminus K; \mathbb{R}) \rightarrow H_c^{2n-2}(X; \mathbb{R}) \rightarrow H_c^{2n-2}(K; \mathbb{R}) \cong 0$$

shows that  $\dim_{\mathbb{R}} H_c^{2n-2}(X; \mathbb{R}) \leq \dim_{\mathbb{R}} H_c^{2n-3}(K; \mathbb{R})$ . Thus,  $\dim_{\mathbb{R}} H_c^{2n-2}(X; \mathbb{R})$  is zero for all  $n \geq 3$  and does not exceed one for  $n = 2$ . In the latter case, however, it follows from Euler characteristic additivity that the dimension of  $H_c^2(X; \mathbb{R})$  is even and hence also equals zero.  $\square$

**6. Symplectic rigidity. Proof of the main result.** If the surgery in Lemma 8 were symplectic (i.e., there were a symplectic form on  $X$  restricting to  $\omega_0$  on  $U$  and  $\mathbb{C}^n \setminus \overline{U}$ ), then the conclusions of Lemmas 8 and 9 for an even  $n$  would contradict the following result:

**Theorem 10** (Eliashberg–Floer–McDuff [12], [3]). *Let  $(X, \omega)$  be a symplectic manifold symplectomorphic to  $(\mathbb{R}^{2n}, \omega_0)$ ,  $n \geq 2$ , outside of a compact subset. Assume that  $[\omega]$  vanishes on all spherical elements in  $H_2(X; \mathbb{R})$ . Then  $X$  is diffeomorphic to  $\mathbb{R}^{2n}$ .*

**Remark 11.** If  $n = 2$ , then  $X$  is actually symplectomorphic to  $(\mathbb{R}^4, \omega_0)$  by Gromov’s classical result [7]. Note, however, that we only need to know that  $X$  must have the  $\mathbb{Z}/2$ -homology of the ball, which is proved in all dimensions by a basic application of pseudoholomorphic curves (see [12], §3.8).

Thus, to prove the main theorem it remains to show that for a *Lagrangian* embedding of the Klein bottle  $K$  there exists a symplectic surgery having property (6). This can be done in all dimensions by the following elementary construction.

Represent  $K$  as the quotient of  $\mathbb{R}^n \setminus \{0\}$  by the  $\mathbb{Z}$ -action generated by the transformation

$$x \longmapsto 2\sigma(x), \quad (7)$$

where  $\sigma \in O_-(\mathbb{R}^n)$  is a reflection (in particular,  $\sigma = \sigma^T = \sigma^{-1}$ ). The cotangent bundle  $T^*K$  is the quotient of  $T^*(\mathbb{R}^n \setminus \{0\}) \cong (\mathbb{R}_x^n \setminus \{0\}) \times \mathbb{R}_y^n$  by the  $\mathbb{Z}$ -action generated by

$$(x, y) \longmapsto (2\sigma(x), \frac{1}{2}\sigma(y)). \quad (8)$$

Note that the Riemannian metric  $g = \frac{1}{\|x\|^2} \sum dx_j^2$  on  $\mathbb{R}^n \setminus \{0\}$  is invariant with respect to (7) and equip  $K$  with the induced metric. Note further that the unit sphere bundle  $ST^*(\mathbb{R}^n \setminus \{0\}) \subset T^*(\mathbb{R}^n \setminus \{0\})$  with respect to  $g$  is the hypersurface  $\{\|y\|^2 = 1/\|x\|^2\}$ .

On  $T^*(\mathbb{R}^n \setminus \{0\})$  with the zero section removed, consider the map

$$(x, y) \longmapsto (-y, x). \quad (9)$$

Obviously, this map preserves the unit sphere bundle  $ST^*(\mathbb{R}^n \setminus \{0\})$  and the canonical symplectic form on  $T^*(\mathbb{R}^n \setminus \{0\})$ . Furthermore, it maps the orbits of the action (8) into orbits. Hence, it defines a symplectomorphism of  $T^*K$  with the zero section removed that maps  $ST^*K$  into itself.

Let us check that the action of the map (9) on  $H_{n-1}(ST^*K; \mathbb{Z}/2)$  satisfies condition (6). The fibre class  $[\delta]$  is represented by the ‘vertical’  $(n-1)$ -sphere  $\{x = \text{const}, \|y\| = 1\}$  and its image is obviously the class of the ‘horizontal’  $(n-1)$ -sphere  $\{\|x\| = 1, y = \text{const}\}$ . Choose  $m = \{\|x\| = 1\} \subset K$  and  $\nu_{m,K}(x) = x$ . For any almost complex structure on  $T^*K$  compatible with the canonical symplectic form, the isotopy class of the  $\mathbb{C}$ -normal pushoff  $m^\sharp = m + J\nu_{m,K}$  is the same as for the standard complex structure, i.e., it is given by the ‘diagonal’  $(n-1)$ -sphere  $\{y = x, \|x\| = 1\} \subset ST^*K$ . It follows immediately that the image of  $[\delta]$  with respect to (9) is  $[\delta] + [m^\sharp]$ , as required.

Finally, if  $K$  is an embedded Lagrangian Klein bottle in a symplectic manifold, we can identify its closed tubular neighbourhood  $\overline{U}$  with the unit disc bundle  $DT^*K$  by a conformally symplectic diffeomorphism and define the gluing map  $f : \partial U \rightarrow \partial U$  as the restriction of the symplectomorphism constructed above to  $ST^*K$ .  $\square$

**Remark 12.** Replacing the action (7) by  $x \mapsto 2x$ , one obtains a completely analogous surgery construction for the product  $S^{n-1} \times S^1$ . Further symplectic surgeries along a Lagrangian Klein bottle or  $S^{n-1} \times S^1$  can be defined by taking the gluing map from the group generated by the map  $f$  induced by (9) and the map  $\tau$  induced by the co-differential of the topologically non-trivial  $g$ -isometry  $x \mapsto \frac{x}{\|x\|^2}$ .

**Remark 13** (Comparison with Luttinger surgery). (i) In the case of the product  $S^{n-1} \times S^1$ , the surgeries found by Luttinger [11] for  $n = 2$  and by Borrelli [1] for  $n = 4$  and  $n = 8$  correspond to the gluing maps  $(f \circ \tau)^k$ , where  $k \in \mathbb{Z}$  and the maps  $f$  and  $\tau$  are defined as in Remark 12. (ii) The surgery used in [15] in the case of the usual Klein bottle  $K^2$  corresponds to the gluing map  $(f \circ \tau)^{-1}$ . In the notation of [15], one has

$$f(\varphi, \psi, \theta) = (-\varphi, \psi + \theta + \pi, -\theta - \pi) \quad \text{and} \quad \tau(\varphi, \psi, \theta) = (-\varphi, \psi, -\theta - \pi)$$

so that  $f \circ \tau = f_{0,-1}$ . (iii) There is an alternative description of these surgeries in terms of regluing Lefschetz pencils via fibrewise symplectic Dehn twists (see the first draft of this paper, [arxiv:0712.1760v1](#), and the references therein).

**Remark 14** (Totally real embeddings). It is perhaps worth mentioning that totally real embeddings  $K^n \hookrightarrow \mathbb{C}^n$  exist for all  $n$ . Indeed, Lalonde [10] constructed Lagrangian immersions  $K^n \looparrowright \mathbb{C}^n$  that are regularly homotopic to embeddings. The existence of totally real embeddings follows in this situation from Gromov's  $h$ -principle (see, e. g., [5], §19.3).

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